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Dynamics of Differential Rotating Sphere and Pulsar

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Abstract

The dynamics of the differential rotating sphere is worked out. In view of the results, a theory for producing the pulse of pulsars is proposed.

1. *Introduction*

The pulsars are believed to be rapidly rotating neutron stars as proposed by Thomas Gold and Jeremiah Ostriker. But the origin of the pulses of the pulsars are not satisfactorily explained. An explanation is proposed in Section 5.

Although the pulses of the pulsars are radio emission, the support for the radio emission is the matter. If the density on some portion of the surface of the pulsars is higher than the rest, and the matter is subjected to a sudden increase in velocity there, then the radio emission there is stronger than the rest of the surface. Hence the pulse is due to the density bands on the surface of the pulsars as concluded from the mathematical solution in the following sections.

2. *Physical Model*

We consider a spherically symmetrical distribution of matter rotating counterclockwise around the north pole. The matter is moving up and down radially because of its temperature distribution.

(Π, Θ, Φ) denotes the velocity in spherical coordinate (r, θ, ϕ) rotating around the z axis at an angular velocity Φ_0, where Φ_0 is the angular velocity of the surface of the sphere as shown in Figure 1. We have

$$\Pi \neq 0$$

and $\qquad (2.1)$

$$\Theta \approx 0$$

since Φ is chosen to be in the direction of rotation.

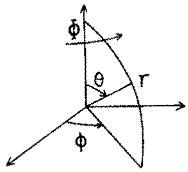

Figure 1.

We assume that the spherical symmetry is not destroyed drastically in the course of time, i.e., the physical model is still more or less spherical in shape. We are interested in the density variation on the outer fringe of this rotating sphere. Hence the potential B on the outer fringe can be approximated by

$$B \approx -GM/r \tag{2.2}$$

where $M \approx$ the total mass.

3. *Mathematical Solution*

$\psi(r, \theta, \phi, \Pi, \Theta, \Phi, t)$ denotes the distribution function of the physical model in Section 2. The Liouville equation in spherical coordinate can be obtained from Chandrasekhar's equation (3.26) (Chandrasekhar, 1960):

$$\begin{aligned} &\frac{\partial \psi}{\partial t} + \Pi \frac{\partial \psi}{\partial r} + \frac{\Theta}{r} \frac{\partial \psi}{\partial \theta} + \frac{\Phi}{r \sin \theta} \frac{\partial \psi}{\partial \phi} + \left(\frac{\Theta^2 + \Phi^2}{r} - \frac{\partial B}{\partial r} \right) \frac{\partial \psi}{\partial \Pi} \\ &\quad + \left(\frac{-\Pi\Theta}{r} + \frac{\cos \theta}{r \sin \theta} \Phi^2 - \frac{1}{r} \frac{\partial B}{\partial \theta} \right) \frac{\partial \psi}{\partial \Phi} \\ &\quad + \left(\frac{-\Pi\Phi}{r} - \frac{\cos \theta}{r \sin \theta} \Theta\Phi - \frac{1}{r \sin \theta} \frac{\partial B}{\partial \phi} \right) \frac{\partial \psi}{\partial \Phi} = 0 \end{aligned} \tag{3.1}$$

Applying (2.1) and (2.2) to (3.1), we have

$$\frac{\partial \psi}{\partial t} + \Pi \frac{\partial \psi}{\partial r} + \frac{\Phi}{r \sin \theta} \frac{\partial \psi}{\partial \phi} + \left(\frac{\Phi^2}{r} - \frac{\partial B}{\partial r} \right) \frac{\partial \psi}{\partial \Pi} - \frac{\Pi\Phi}{r} \frac{\partial \psi}{\partial \Phi} = 0 \tag{3.2}$$

The density of the sphere is given by

$$\sigma(r, \theta, \phi, t) = \iint_{-\infty}^{\infty} \psi \, d\Pi \, d\Phi \tag{3.3}$$

The characteristic equation for (3.2) is

$$\frac{dt}{1} = \frac{dr}{\Pi} = \frac{d\theta}{0} = \frac{d\phi}{\Phi/r \sin \theta} = \frac{d\Pi}{\Phi^2/r - \partial B/\partial r} = \frac{d\Phi}{-\Pi\Phi/r} \tag{3.4}$$

From the third equation in (3.4), we have

$$\theta = \text{const} \tag{3.5}$$

From the last two equations in (3.4) and the time independent of B, the energy integral is obtained:

$$\tfrac{1}{2}(\Pi^2 + \Phi^2) - GM/r = E \tag{3.6}$$

where E is a constant.

From the second and the last equations in (3.4), we obtain the momentum integral:

$$r\Phi = J \tag{3.7}$$

where J is a positive constant.

From the fourth and the last equations in (3.4), we have

$$\frac{\sin\theta\, d\phi}{1} = \frac{d\Phi}{-\Pi}$$

Solving Π from (3.6) and substituting into the above equation and integrating, we obtain

$$\Pi \sin(\phi \sin\theta) + \Phi \cos(\phi \sin\theta) - (GM/J)\cos\theta = [2E + (GM/J)^2]^{1/2} \sin C_1 \tag{3.8}$$

Dividing by Φ and using (3.7), we obtain

$$\Pi/\Phi = \cot(\phi \sin\theta) + (2E/J)\,\xi(r, \phi, \theta) \tag{3.9}$$

where

$$\xi(r, \phi, \theta) = \frac{GM}{2JE} r \cot(\phi \sin\phi) + \frac{[2E + (GM/J)^2]^{1/2}}{2E} r \sin C_1 \operatorname{cosec}(\phi \sin\theta) \tag{3.10}$$

where C_1 is a constant.

From the first and the second equations in (3.4), we have

$$\frac{dt}{1} = \frac{dr}{\Pi}$$

Applying (3.6) and (3.7) and integrating, we obtain

$$t = \frac{-J}{2E}\frac{\Pi}{\Phi} + \lambda(r) + C_2 \tag{3.11}$$

where

$$\lambda(r) = \frac{-GM}{(2E)^{3/2}} \ln |\, 2(2E)^{1/2}(2Er^2 + 2GMr - J^2)^{1/2} + 4Er + 2GM\, | \tag{3.12}$$

Substituting (3.9) into (3.11), we obtain

$$t = (-J/2E)\cot(\phi \sin\theta) - \xi(r, \phi, \theta) + \lambda(r) + C_2 \tag{3.13}$$

There the general solution for (3.2) is

$$\psi = \psi[(\Pi^2 + \Phi^2)/2 - GM/r, r\Phi, \Pi \sin(\phi \sin\theta) + \Phi \cos(\phi \sin\theta)$$
$$-(GM/J)\cos(\phi \sin\theta), t + (J/2E)\cot(\phi \sin\theta) + \xi(r, \phi, \theta) - \lambda(r)] \tag{3.14}$$

The density is

$$\sigma(r, \theta, \phi, t) = A(r, \theta, \phi)\cos[t + (J/2E)\cot(\phi \sin\theta) + \xi(r, \phi, \theta) - \lambda(r)] \tag{3.15}$$

where the amplitude $A(r, \theta, \phi)$ can be obtained from (3.14).

4. *Physical Interpretation*

We set

$$\chi = t + (J/2E)\cot(\phi \sin\theta) + \xi(r, \theta, \phi) - \lambda(r) \tag{4.1}$$

For practical values of r, J, E and M, we have

$$\chi \sim t + (J/2E)\cot(\phi \sin\theta) \tag{4.2}$$

From (3.15), we know the motion of the density wave is determined by $\cos\chi$. The peaks of the density waves occur at

$$t + \frac{J}{2E}\cot(\phi \sin\theta) = n\pi \tag{4.3}$$

The phase velocity in the ϕ direction is

$$\frac{d\phi}{dt} = \frac{2E}{J}\frac{\sin^2(\phi \sin\theta)}{\sin\theta} \tag{4.4}$$

We see that $\chi = \infty$ or $-\infty$ and $d\phi/dt = 0$ when $\phi = 0$ or $\phi \sin\theta = \pi$. Let us denote the curves $\phi = 0$ and $\phi \sin\theta = \pi$ by B_1 and B_2, respectively. Because of the factor $\sin\theta$, the middle point of B_2 starts from $\phi = \pi$, $\theta = \pi/2$ and runs up and down in the ϕ direction and meets B_1 at $\theta = 30°$ or $150°$ as shown in Figure 2. Since χ tends to $\pm\infty$ as ϕ tends to B_1 or B_2, there will

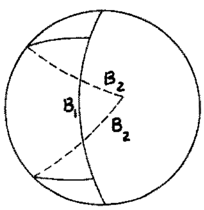

Figure 2.

be many density-wave peaks crowding around B_1 or B_2, respectively. The spacing between the peaks is smaller as ϕ tends to B_1 or B_2. The phase velocity in the ϕ direction is faster as ϕ is farther away from B_1 or B_2.

Therefore the density waves will pile up on one side and split up on the other side of B_1 and B_2. Sharp density shocks, B_1 and B_2, will be produced.

5. *Pulsar*

Since it is believed that the interior of the pulsars is viscousless, the rapidly rotating neutron star model for pulsars fits very well with the physical model described in Section 2. The mathematical results and the physical interpretation in Sections 3 and 4 indicate that density waves will be formed automatically. The density waves will pile up along two bands, B_1 and B_2. Furthermore, the particles are subjected to sudden acceleration and deceleration around the bands. The neutron star radiates radio waves in all directions, but the radio waves are stronger on the sharp density bands B_1 and B_2. Hence we receive pulses from the rapidly rotating neutron stars. Since the second band B_2 is weaker than the first band B_1, the pulse from the band B_2 is weaker than that from the first band B_1. This explains one stronger pulse and one weaker and off center pulse of the pulsars.

References

Chandrasekhar, S. (1960). *Principles of Stellar Dynamics.* (Dover, New York).
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